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Closed-Loop Pole Design for Vibration Suppression

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A technique is discussed for selecting the closed-loop pole locations in the state or output feedback problem. The damping is the only parameter that is allowed to vary. The geometric interpretation of this is that each closed-loop pole is constrained to lie on a circular arc whose radius corresponds to that pole's open-loop undamped natural frequency. An analytic approach is then proposed to compute the required state and output feedback gain. The method is based on a sensitivity analysis of the closed-loop eigenvalues to each gain element. An Euler–Bernoulli pinned beam example is used to demonstrate the procedure. This new formulation offers insight into the uniqueness issue in state design, the possibility of state eigenstructure assignability, and the limited pole placement in output design from a linear algebra context.

Nomenclature

A	= continuous system matrix
B	= continuous input matrix
C	= output matrix
K, K_y	= output feedback gain matrix
K_x	= state feedback gain matrix
k_{ij}	= ijth gain element
m	= number of inputs
n	= system order
r	= number of outputs
s_h	= change in hth eigenvalue (or eigenvector) from all ij gain elements
s_h^{ij}	= change in hth eigenvalue (or eigenvector) from ijth gain element
$\boldsymbol{u}$	= control input vector
$\boldsymbol{v}_h$	= hth right eigenvector
$\boldsymbol{w}_h$	= hth left eigenvector
$\boldsymbol{x}$	= system state vector
$\boldsymbol{y}$	= output vector
α	= percent damping parameter (>0)
Δk_{ij}	= ijth gain element increment
Λ_{closed}	= diagonal matrix of closed-loop poles
λ	= complex eigenvalue
ξ	= modal damping
ω_d	= damped natural frequency
ω_n	= undamped natural frequency
$\bar{\omega}_n$	= new undamped natural frequency

Introduction

A NUMBER of schemes to locate the closed-loop poles, whether for state or output design, can be found in Refs. 1–3. Whereas these schemes selectively place the poles in prespecified regions of the complex plane, the necessary gain matrix can still be poorly conditioned, provided one exists. Furthermore, those schemes alter the physical nature of the structure. Consider the vibration suppression problem, for example. It is understood that the closed-loop poles should be located deeper in the left-half complex plane for increased stability. However, it may sometimes be unlikely that a system's natural frequency can be significantly altered through some electromechanical means (i.e., from a controller). That is, the actuator can saturate if one attempts to change the natural frequency in a structural-control application. This can be the case since altering the natural frequency often requires high control forces. Furthermore, high forces are typically undesirable since they lead to large actuators and high power requirements and can cause spillover instability.

Meirovitch[4] points out that it is not necessary to alter the natural frequency to guarantee asymptotic stability. This paper adopts a similar view and does not alter the system's natural frequencies. It is shown that this leads to placing each closed-loop pole on a circle whose radius corresponds to that pole's open-loop undamped natural frequency. Stability is then guaranteed by remaining between the open-loop complex conjugate poles. Because the frequency is to remain constant, the damping is the only parameter that can be affected. From a fundamental viewpoint, it should be the damping that needs to increase to suppress unwanted vibrations. An alternative concept along these lines is to fix the damped natural frequency. This is perhaps more intuitive because an underdamped structure will oscillate at its damped natural frequency as opposed to its undamped natural frequency.

Once selecting the desired pole locations, the required gain matrix is then sought. There are several methods that can be used to approach this problem. Among them is optimal control theory and the so-called linear quadratic regulator (LQR) problem.[5–8] The LQR strategy minimizes the weighted sum of the state and control cost. Although the solution is optimal, there are some drawbacks. One needs to specify the particular weight matrices and the system may still possess an insufficient response time. To circumvent this, Solheim,[9] Luo and Lan,[10] and Juang and Lee[11] derive a technique whereby the weights are determined from knowledge of the desired closed-loop poles. An optimization approach using genetic algorithms can also be used for control system design and are found in the works of Krishnakumar and Goldberg[12] and Porter and Borair,[13] for instance.

This paper compares the proposed analytical formulation to results obtained from standard algorithms such as the Kautsky et al.[14] method and the Simon–Mitter[15] method. Furthermore, the proposed method addresses the uniqueness issue in state design, the possibility of full state eigenstructure (i.e., eigenvalue and eigenvector) assignability, and reveals the limited pole assignability in the output design.

Problem Formulation

Consider an n-dimensional, linear, time-invariant, r output, m input continuous system represented as

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \qquad \boldsymbol{y}(t) = C\boldsymbol{x}(t) \tag{1}$$

where $R^{n\times n}$ denotes the set of real $n \times n$ matrices, $\boldsymbol{x} \in R^{n\times 1}$, $\boldsymbol{y} \in R^{r\times 1}$, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{r\times n}$, and $\boldsymbol{u} \in R^{m\times 1}$. The system states in Eq. (1) are given by $\boldsymbol{x}$, whereas $\boldsymbol{u}$ is the control force and $\boldsymbol{y}$ are the outputs. For most practical structural-control applications, n will be even and represent twice the number of vibration modes for an underdamped system.

The control force in the output feedback problem is assumed to be a linear combination of the outputs, i.e.,

$$\boldsymbol{u}(t) = -K\boldsymbol{y}(t) = -KC\boldsymbol{x}(t) \tag{2}$$

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where $K \in R^{m \times r}$. Substitution of Eq. (2) into the state of Eq. (1) then results in the output feedback closed-loop system described by

$$\dot{x}(t) = (A - BKC)x(t) \tag{3}$$

The state feedback case is a specific form of Eq. (3) with $C = I_{n \times n}$, where I denotes the identity matrix of order n. The problem then becomes one of selecting the gain matrix K such that

$$\sigma(A - BKC) = \Lambda_{\text{closed}} \tag{4}$$

where $\sigma(\cdot)$ denotes the eigenvalues of $(\cdot)$ and Λ_{closed} is a prescribed diagonal matrix of closed-loop poles. Although there are several ways to arrive at an appropriate Λ_{closed}, it is our intent to select the diagonal elements of Λ_{closed} in a specific manner.

It is clear that the open-loop poles of Eq. (1) are determined from an eigenanalysis. Now, plot the open-loop poles in the complex plane and move each pole along a circular arc whose radius is defined as the undamped natural frequency of that pole. Figure 1 depicts the situation for a single open-loop pole. The dashed line represents the loci of points with constant undamped natural frequency. Each closed-loop pole is then selected to lie on the circular arc. The eigenvalue is given as $\lambda = -\xi\omega_n \pm i\omega_d$, and ξ, ω_n, and ω_d are the damping, undamped natural frequency, and damped natural frequency, respectively. It is obvious that to stay on the circle defined by the open-loop natural frequency, the damping is the only parameter that can be changed. Therefore, the open- and closed-loop eigenvalues can be written as

$$\lambda_{\text{open}} = -\xi_{\text{open}}\omega_n^{\text{open}} \pm i\omega_d^{\text{open}} \tag{5}$$

$$\lambda_{\text{closed}} = -\xi_{\text{closed}}\omega_n^{\text{open}} \pm i\omega_d^{\text{closed}} \tag{6}$$

where $\omega_d = \omega_n\sqrt{(1 - \xi^2)}$. Thus, the only unknown in Eq. (6) is ξ_{closed}, which can be obtained from

$$\xi_{\text{closed}} = \begin{cases} [1 + (\alpha/100)]\xi_{\text{open}} & \text{for} \quad \xi_{\text{open}} \neq 0 \\ \alpha/100 & \text{for} \quad \xi_{\text{open}} = 0 \end{cases} \tag{7}$$

where $\alpha \geq 0$ represents the percentage increase in the open-loop damping. For example, $\alpha = 10$ corresponds to a 10% damping increase, whereas $\alpha = 100$ gives 100%, and so forth. Application of Eqs. (6) and (7) for all n open-loop eigenvalues and arranging them into a diagonal matrix will produce the desired Λ_{closed} matrix.

Instead of fixing the undamped natural frequency, it may be more appealing to fix the damped natural frequency because structures vibrate with their damped frequencies. This approach is also shown in Fig. 1 where the open-loop pole is simply moved horizontally. The equation that governs this is

$$\omega_d^{\text{open}} = \omega_d^{\text{closed}} \tag{8}$$

or

$$\bar{\omega}_n = \frac{\omega_n^{\text{open}}\sqrt{\left(1 - \xi_{\text{open}}^2\right)}}{\sqrt{\left(1 - \xi_{\text{closed}}^2\right)}} \tag{9}$$

where $\bar{\omega}_n$ is the new undamped natural frequency, from which the closed-loop eigenvalues are given in Eq. (6) with $\bar{\omega}_n$ replacing ω_n^{open}. For lowly damped systems it is recognized that Eqs. (6) and (7) and Eqs. (6) and (9) produce almost identical closed-loop eigenvalues.

It will be tacitly assumed herein that the hth eigenvalue of Eq. (3) is a differentiable function of the ijth gain element (assumption 1). Furthermore, all eigenvalues are assumed to occur in complex conjugate pairs (assumption 2). Given assumption 1, a first-order eigenvalue Taylor series expansion yields

$$\lambda_h(k_{ij} + \Delta k_{ij}) = \lambda_h(k_{ij}) + \left.\frac{\partial \lambda_h}{\partial k_{ij}}\right|_{k_{ij}} \Delta k_{ij}$$

$$i = 1, \ldots, m, \quad j = 1, \ldots, r, \quad h = 1, \ldots, n \tag{10}$$

where k_{ij} is the starting ijth gain and Δk_{ij} is the incremental change in the ijth gain. The change in the hth eigenvalue due to a variation in the ijth gain (i.e., s_h^{ij}) is then represented as

$$s_h^{ij} = \lambda_h(k_{ij} + \Delta k_{ij}) - \lambda_h(k_{ij}) = \left.\frac{\partial \lambda_h}{\partial k_{ij}}\right|_{k_{ij}} \Delta k_{ij} \tag{11}$$

Note that Eq. (11) is the change due to a single gain element. It is of interest to compute the change from all gain elements on the hth eigenvalue, s_h,

$$s_h = \sum_{i=1}^{m}\sum_{j=1}^{r} s_h^{ij} = \sum_{i=1}^{m}\sum_{j=1}^{r} \left.\frac{\partial \lambda_h}{\partial k_{ij}}\right|_{k_{ij}} \Delta k_{ij}, \qquad h = 1, \ldots, n \tag{12}$$

Equation (12) is the key equation, and it relates the hth eigenvalue change to the gain increments. The following definition is then made:

$$a_{ij}^h = \left.\frac{\partial \lambda_h}{\partial k_{ij}}\right|_{k_{ij}} \tag{13}$$

In general, Eq. (12) represents a complex equation and, hence, the real and imaginary components can be separated as

$$\text{Re}\, s_h = \sum_{i=1}^{m}\sum_{j=1}^{r} \text{Re}\left(a_{ij}^h\right)\Delta k_{ij} \tag{14}$$

$$\text{Im}\, s_h = \sum_{i=1}^{m}\sum_{j=1}^{r} \text{Im}\left(a_{ij}^h\right)\Delta k_{ij} \tag{15}$$

Let us make the following observation. There are obviously a total of n complex open-loop eigenvalues with each eigenvalue possessing a complex conjugate. Therefore, there exist only $n/2$ distinct complex eigenvalues, and because each distinct complex eigenvalue has a real and imaginary component, there will be $2 \times (n/2) = n$ equations from Eqs. (14) and (15). It is assumed that a change for one complex eigenvalue will automatically occur for its conjugate (hence the need for assumption 2). Then, considering only the $n/2$ distinct complex eigenvalues, the matrix equivalent of Eqs. (14) and (15) is

$$\underset{n \times 1}{\begin{Bmatrix} \text{Re}\, s_1 \\ \vdots \\ \text{Re}\, s_{n/2} \\ \text{Im}\, s_1 \\ \vdots \\ \text{Im}\, s_{n/2} \end{Bmatrix}} = \underset{n \times mr}{\begin{bmatrix} \text{Re}\, a_{11}^1 & \cdots & \text{Re}\, a_{1r}^1 & \text{Re}\, a_{21}^1 & \cdots & \text{Re}\, a_{2r}^1 & \cdots & \text{Re}\, a_{m1}^1 & \cdots & \text{Re}\, a_{mr}^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{Re}\, a_{11}^{n/2} & \cdots & \text{Re}\, a_{1r}^{n/2} & \text{Re}\, a_{21}^{n/2} & \cdots & \text{Re}\, a_{2r}^{n/2} & \cdots & \text{Re}\, a_{m1}^{n/2} & \cdots & \text{Re}\, a_{mr}^{n/2} \\ \text{Im}\, a_{11}^1 & \cdots & \text{Im}\, a_{1r}^1 & \text{Im}\, a_{21}^1 & \cdots & \text{Im}\, a_{2r}^1 & \cdots & \text{Im}\, a_{m1}^1 & \cdots & \text{Im}\, a_{mr}^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{Im}\, a_{11}^{n/2} & \cdots & \text{Im}\, a_{1r}^{n/2} & \text{Im}\, a_{21}^{n/2} & \cdots & \text{Im}\, a_{2r}^{n/2} & \cdots & \text{Im}\, a_{m1}^{n/2} & \cdots & \text{Im}\, a_{mr}^{n/2} \end{bmatrix}} \underset{mr \times 1}{\begin{Bmatrix} \Delta k_{11} \\ \vdots \\ \Delta k_{1r} \\ \Delta k_{21} \\ \vdots \\ \Delta k_{2r} \\ \vdots \\ \Delta k_{m1} \\ \vdots \\ \Delta k_{mr} \end{Bmatrix}} \tag{16}$$

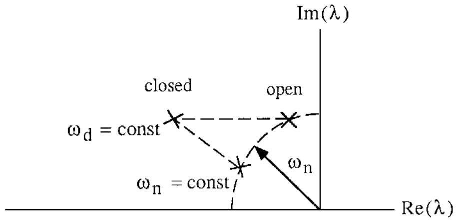

Fig. 1 Open- and closed-loop poles (upper half shown only).

Equation (16) clearly depicts a set of n simultaneous linear equations, which can be easily solved. It remains only to compute the left-hand side and the matrix of a_{ij}^h. The values of s_h are simply computed from $s_h = \lambda_h^{\text{closed}} - \lambda_h^{\text{open}}$, $h = 1, \ldots, n/2$, and a_{ij}^h is obtained from the following theorem.[16]

Eigenvalue sensitivity theorem. Given the dynamic system in Eq. (3), the sensitivity of the hth eigenvalue to changes in the ijth element of K is

$$\frac{\partial \lambda_h}{\partial k_{ij}} = \frac{-\boldsymbol{w}_h^T b_i c_j \boldsymbol{v}_h}{\boldsymbol{w}_h^T \boldsymbol{v}_h} \tag{17}$$

where $\boldsymbol{w}_h$ and $\boldsymbol{v}_h$ are the right eigenvectors of $(A - BKC)^T$ and $(A - BKC)$, respectively, b_i is the ith column of B, c_j is the jth row of C, and superscript T denotes a transpose.

Observe that Eq. (16) applies to both the state and output feedback problems. In addition, it is known that a unique state feedback gain exists for a single-input system and a nonunique gain exists for a multi-input system. Masui et al.[17] show this by considering an extended system where the inputs are considered as state variables. This fact is easily demonstrated directly from Eq. (16). It is clearly seen that there are n equations for m^*r unknowns. In state feedback, $r = n$. Consequently, there are n equations and m^*n unknowns. A unique solution exists only if $m = 1$ (single input) because there are as many equations as unknowns. A nonunique solution exists in the multi-input case ($m > 1$) because there are more equations than unknowns. In addition, Eq. (16) also admits the solution of the state eigenstructure assignability. That is, to place not only the eigenvalues but eigenvectors as well. Consider a set of $n/2$ distinct complex closed-loop eigenvectors from $\boldsymbol{v}_h^{\text{closed}}$, $h = 1, \ldots, n$. Taking a similar sensitivity approach as for the eigenvalues, there results

$$s_h = \sum_{i=1}^{m} \sum_{j=1}^{r} \left. \frac{\partial \boldsymbol{v}_h}{\partial k_{ij}} \right|_{k_{ij}} \Delta k_{ij}, \qquad h = 1, \ldots, n \tag{18}$$

The eigenvector change can be computed from $s_h = \boldsymbol{v}_h^{\text{closed}} - \boldsymbol{v}_h^{\text{open}}$, $h = 1, \ldots, n/2$, and the eigenvector derivatives can be obtained from the following theorem.[18]

Eigenvector sensitivity theorem. Given the dynamic system in Eq. (3), the sensitivity of the hth eigenvector to changes in the ijth element of K is

$$\frac{\partial \boldsymbol{v}_h}{\partial k_{ij}} = \sum_{m=1}^{n} \alpha_{ijhm} \boldsymbol{v}_m \tag{19}$$

where

$$\alpha_{ijhq} = \frac{\boldsymbol{w}_q^T b_i c_j \boldsymbol{v}_h}{(\lambda_q - \lambda_h) \boldsymbol{w}_q^T \boldsymbol{v}_q}, \qquad q \neq h$$

$$\alpha_{ijhh} = -\frac{1}{\boldsymbol{v}_h^T \boldsymbol{v}_h} \sum_{\substack{m=1 \\ m \neq h}}^{n} \alpha_{ijhm} \boldsymbol{v}_m^T \boldsymbol{v}_h, \qquad q = h$$

A multi-input system ($m > 1$) will then yield a unique solution if an additional $n^*(m - 1)$ equations can be found. Therefore, only $[n^*(m - 1)]/2$ eigenvector components can be placed. Also, note that for a nonunique solution the minimum norm gain is easily computed from a pseudoinverse calculation.[19] Two advantages of the minimum norm gain is the small control forces and the possibility of reducing the effect of spillover.[20,21] The case of output feedback design is readily apparent. In this case, there will be more equations (n) than unknowns (m^*r) and so complete eigenvalue freedom is not possible. However, Eq. (16) can still be solved such that the residual is minimized in a least squares sense.

This eigenvalue sensitivity approach to the state feedback problem is denoted as the analytical sensitivity formulation (ASF) method. Application of Eq. (16) to the output feedback problem will be referred to as a direct output ASF solution method. An indirect output approach is also suggested in the work of Munro and Vardulakis,[22] from which the following theorem is stated.

Theorem 1. A necessary and sufficient condition for placement of all of the poles in the output feedback system is that the state feedback matrix K_x satisfies

$$K_x C^{g_1} C = K_x \tag{20}$$

where C^{g_1} is a g_1 inverse of C. Under these conditions, the required output feedback matrix K_y becomes

$$K_y = K_x C^{g_1} \tag{21}$$

The procedure then is to solve the state feedback problem for K_x and use Eq. (21) to identify the output feedback matrix K_y. The g_1 inverse of C can be taken as the right inverse of Penrose (see Ref. 22), $C^{g_1} = C^T(CC^T)^{-1}$. It is realized, though, that Theorem 1 can never be satisfied in a structural-control application. The point is that the result of the theorem can still be used. The reason is that Eq. (21) represents the best possible solution, in a least squares sense, because the following equation needs to be satisfied: $K_y C = K_x$. This approach was also suggested by Balas[23] and will be referred to as the indirect output method.

Finally, it was implicitly assumed that the proposed ASF method be applied in a one-step solution format. That is, the closed-loop poles are reached with one iteration from the starting open-loop poles. If the computed closed-loop eigenvalues are not acceptable, an iterative marching technique can be used where the open-loop damping is increased incrementally until the desired set of closed-loop poles is reached. Furthermore, the ASF methodology is also applicable in a Luenberger observer design.

Results

State and output feedback designs were implemented on the Euler–Bernoulli pinned beam example of Balas.[23] The first three modes were taken as the controlled modes of the system and are given by natural frequency $\omega_k = (k\pi)^2$ and mode shape $\phi_k(\boldsymbol{x}) = \sin(k\pi \boldsymbol{x})$. The system state-space model is given as

$$A = \begin{bmatrix} 0_3 & I_3 \\ -\Lambda^2 & 0_3 \end{bmatrix} \qquad B = \begin{bmatrix} 0_{3\times 2} \\ B_n \end{bmatrix}$$

$$B_n = \begin{bmatrix} \phi_1\left(\frac{1}{5}\right) & \phi_1\left(\frac{4}{5}\right) \\ \phi_2\left(\frac{1}{5}\right) & \phi_2\left(\frac{4}{5}\right) \\ \phi_3\left(\frac{1}{5}\right) & \phi_3\left(\frac{4}{5}\right) \end{bmatrix} = \begin{bmatrix} 0.59 & 0.59 \\ 0.95 & -0.95 \\ 0.95 & 0.95 \end{bmatrix} \qquad C = B^T$$

where $\Lambda^2 = \text{diag}(\omega_1^2, \omega_2^2, \omega_3^2)$. This system represents an actuator-velocity sensor pair at $\frac{1}{5}$ and $\frac{4}{5}$ of the beam length. Matrices A and B are used in the state feedback problem, whereas matrix C is used in the output feedback problem. The desired closed pole locations were selected as $-0.99 \pm i9.8$, $-3.9 \pm i39.3$, and $-8.8 \pm i88.4$. These poles represent the closed-loop frequency and damping as given in Table 1, which maintains a constant undamped natural frequency.

Table 1 Open- and desired closed-loop beam behavior

Open loop		Desired closed loop	
Freq, Hz	Damp, %	Freq, Hz	Damp, %
1.5708	0	1.5677	10.05
6.2832	0	6.2855	9.88
14.1372	0	14.1388	9.91

Table 2 Comparison of closed-loop frequency and damping for state feedback design

KND		SM		One-step ASF	
Freq, Hz	Damp, %	Freq, Hz	Damp, %	Freq, Hz	Damp, %
1.5677	10.05	1.5708	10.06	1.5632	10.24
6.2855	9.88	6.2829	9.99	6.2547	9.92
14.1388	9.91	14.1372	9.99	14.0369	9.96

Table 3 Closed-loop pole results for the indirect output feedback approach

KND		SM		One-step ASF	
λ_{closed}		λ_{closed}		λ_{closed}	
$-3.4 \pm i9.4$		$-3.3 \pm i9.6$		$-3.2 \pm i9.4$	
$-1.8 \pm i39.4$		$-4.7 \pm i38.8$		$-4.4 \pm i39.2$	
$-8.6 \pm i87.7$		$-7.9 \pm i86.8$		$-8.1 \pm i87.9$	
Freq, Hz	Damp, %	Freq, Hz	Damp, %	Freq, Hz	Damp, %
1.5845	33.94	1.6156	32.51	1.5814	32.15
6.2802	4.45	6.2204	12.03	6.2832	11.26
14.0220	9.71	13.8717	9.06	14.0426	9.22

Balas[23] gives the Simon–Mitter (SM) state feedback solution as

$$K_{SM} = \begin{bmatrix} -0.84 & 18.82 & 73.71 & 1.12 & -8.35 & 6.21 \\ -1.68 & 37.65 & 147.42 & 2.24 & -16.70 & 12.42 \end{bmatrix}$$

whose Frobenius norm is 171.7, whereas the Kautsky et al.[14] (KND) solution is computed as

$$K_{KND} = \begin{bmatrix} 508.06 & -998.15 & -551.36 & 4.59 & 0.22 & 7.65 \\ 1468 & 108.17 & 555.2 & 11.11 & -3.47 & 7.73 \end{bmatrix}$$

whose norm is 2008.4. The KND method is available in MATLAB™ as m-file place.m. However, the one-step ASF state solution is given as

$$K_{ASF} = \begin{bmatrix} -1.16 & -7.41 & -39.87 & 1.68 & 4.11 & 9.26 \\ -1.16 & 7.41 & -39.87 & 1.68 & -4.11 & 9.26 \end{bmatrix}$$

with a norm of 59.2 (a 65.5% reduction over SM and a 97.1% reduction over KND). The closed-loop frequency and damping values are listed in Table 2, which shows a good correlation between the three methods and to the desired behavior in Table 1.

Results from the output feedback problem are now presented. The closed-loop eigenvalues, undamped natural frequencies, and damping are listed in Table 3 using the indirect output approach. That is, the required output feedback gain matrix is computed from Eq. (21). The corresponding output matrix gains were

$$K_{KND} = \begin{bmatrix} 4.10 & 3.87 \\ 3.73 & 7.38 \end{bmatrix} \qquad \text{norm} = 10.01$$

$$K_{SM} = \begin{bmatrix} -2.02 & 8.02 \\ 4.03 & 16.04 \end{bmatrix} \qquad \text{norm} = 18.49$$

$$K_{ASF} = \begin{bmatrix} 6.99 & 2.07 \\ 2.07 & 6.99 \end{bmatrix} \qquad \text{norm} = 10.31$$

Whereas the KND solution produced the lowest Frobenius norm, the second mode damping was not as high as with the SM and ASF methods. Furthermore, for almost the same gain norm as KND, the ASF method produced a more stable closed-loop system as seen in the second mode damping. For comparison purposes, the one-step direct output ASF solution is given in Table 4 with the following gain matrix and norm:

$$K_{ASF} = \begin{bmatrix} 6.59 & 2.27 \\ 2.27 & 6.59 \end{bmatrix} \qquad \text{norm} = 9.85$$

whose result is surprisingly good.

Table 4 Closed-loop pole results for the one-step direct output ASF solution

Freq, Hz	Damp, %
1.5809	31.43
6.2832	9.88
14.0468	9.01

Conclusions

An analytical formulation was derived based on a sensitivity calculation to solve the state and output feedback design problems. A direct and indirect solution was discussed for the output feedback case. An Euler–Bernoulli beam example was used to validate the proposed sensitivity technique. This novel formulation produced a more significant gain norm reduction than other eigenvalue placement algorithms, such as the SM method and the KND method. Finally, the proposed design method addressed the uniqueness issue in state design, allowed the possibility of state eigenstructure assignment, and showed the limited pole assignability in the output case.

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